On the mode stability of a self-similar wave map

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We study linear perturbations of a self-similar wave map from Minkowski space to the three-sphere which is conjectured to be linearly stable. Considering analytic mode solutions of the evolution equation for the perturbations we prove that there are no real unstable eigenvalues apart from the well-known gauge instability.

I. INTRODUCTION

A. Motivation

When considering the Cauchy problem for nonlinear evolution equations one often encounters a behaviour known as "blow up": Evolutions with regular initial data become singular after a finite time. Moreover, it depends on the "size" of the data whether the blow up occurs or not. Such phenomena have been observed for many different equations arising from various branches of physics, chemistry, biology, etc. In particular Einstein's equations of general relativity have this property. Furthermore, in many cases there is some kind of universal behaviour in the sense that the shape of the blow up profile is independent of the special form of the data.

Co-rotational wave maps from Minkowski space to the three-sphere have been used as a toy-model for blow up phenomena in general relativity (cf. Ref. 1, Ref. 3). Singularity formation for these wave maps has been studied extensively using numerical techniques but there are very few rigorous results available. In numerical simulations (Ref. 3) one observes that singularity formation takes place in a universal manner via a certain self-similar solution f_0. The existence of f_0 has been established rigorously in Ref. 4 and later it has been found in closed form (Ref. 5). Using adapted coordinates, this self–similar blow up can be reformulated as an asymptotic stability problem. Thus, it is conjectured that the solution f_0 is linearly stable.

However, the eigenvalue equation for perturbation modes around f_0 (although a linear ordinary differential equation of second order) is hard to handle. In particular, it is not possible to analyse it with standard self–adjoint Sturm–Liouville techniques. Therefore, not even mode stability (i.e. the non-existence of eigenvalues with positive real parts) of f_0 has been proved so far. Nevertheless, Sturm–Liouville theory can be applied at least partly and Bizoń was able to show that there are no eigenvalues with real parts greater than 1 (Ref. 2). The main result of the present paper is the non–existence of mode solutions with eigenvalues between 0 and 1 which further supports the conjecture of linear stability of f_0.

B. Derivation of the equations, main results

We study wave maps from Minkowski space to the three-sphere (see Ref. 1 for a definition) and restrict ourselves to co-rotational solutions which reduces the problem to the single semilinear wave

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equation

$$\Psi_{tt}(t,r) = \Psi_{rr}(t,r) + \frac{2}{r}\Psi_r(t,r) - \frac{\sin(2\Psi(t,r))}{r^2}. \tag{1}$$

Since we are interested in self-similar solutions we introduce adapted coordinates $\tau := -\log(T-t)$ and $\rho := r/(T-t)$ where $T > 0$ is an arbitrary constant (the blow up time). The self–similar solution f_0 is given by $f_0(\rho) = 2\arctan(\rho)$. We consider linear perturbations of f_0 and insert the ansatz

$$\Psi(\tau,\rho) = f_0(\rho) + w(\tau,\rho)$$

into eq. (1). Neglecting higher-order terms leads to

$$w_{\tau\tau} - (1-\rho^2)w_{\rho\rho} + 2\rho w_{\tau\rho} + w_\tau - 2\frac{1-\rho^2}{\rho}w_\rho + \frac{2\cos(2f_0)}{\rho^2}w = 0 \tag{2}$$

which is a time evolution equation for linear perturbations w of f_0. The wave map f_0 is considered to be linearly stable if all solutions of eq. (2) remain bounded (with respect to a suitable norm) for all $\tau > 0$.

We are interested in mode solutions of eq. (2), i.e. solutions of the form

$$w(\tau,\rho) = e^{\lambda\tau}u(\rho)$$

for $\lambda \in \mathbb{C}$. Inserting this ansatz into eq. (2) yields

$$u'' + \left(\frac{2}{\rho} - \frac{2\lambda\rho}{1-\rho^2}\right)u' - \left(\frac{2\cos(2f_0)}{\rho^2(1-\rho^2)} + \frac{\lambda(1+\lambda)}{1-\rho^2}\right)u = 0 \tag{3}$$

which is a second-order ordinary differential equation (ODE) for u with singular points at $\rho = 0$ and $\rho = 1$. Thus, a necessary condition for linear stability of f_0 is the non-existence of mode solutions with $\mathrm{Re}\lambda > 0$. It is a subtle question what degree of regularity one should require to consider a solution of eq. (3) as "admissible". However, we will at least assume that any admissible solution should be differentiable on $[0,1]$. Thus, by a regular solution of eq. (3) we mean a continuous function $u : [0,1] \to \mathbb{C}$ that solves eq. (3) on $(0,1)$ in the classical sense and u' can be continuously extended to the whole interval $[0,1]$. We remark that eq. (3) does not constitute a standard eigenvalue problem since the coefficient of the first derivative of u depends on the spectral parameter λ. Nevertheless we will speak of eigenvalues and eigenfunctions. A simple change of the dependent variable $u \mapsto \widetilde{u}$ where

$$\tilde{u}(\rho) := \rho(1-\rho^2)^{\lambda/2}u(\rho)$$

leads to the equation

$$\tilde{u}'' - \left(\frac{2\cos(2f_0)}{\rho^2(1-\rho^2)} - \frac{\lambda(2-\lambda)}{(1-\rho^2)^2}\right)\tilde{u} = 0. \tag{4}$$

Both forms eq. (3) and eq. (4) of the eigenvalue problem will be useful in the sequel. Now we are ready to formulate our result.

Theorem 1. *For $\lambda \in (0,1)$ there does not exist a regular solution of eq. (3).*

C. Notations

We will make frequent use of the following standard notations. By $C^k[a,b]$ we denote the vector space of continuous functions $u : [a,b] \to \mathbb{C}$ which are k–times continuously differentiable on (a,b)

and all k derivatives can be extended to continuous functions on $[a,b]$. On this space we define a norm by

$$\|u\|_{C^k[a,b]} := \sum_{j=0}^{k} \sup_{x\in[a,b]} |u^{(j)}(x)|.$$

The normed vector space $C^k[a,b]$ together with $\|\cdot\|_{C^k[a,b]}$ is a Banach space. We also mention (weighted) Lebesgue spaces

$$L^p((a,b), w(x)dx)$$

defined by (equivalence classes of) functions $u : [a,b] \to \mathbb{C}$ such that

$$\int_a^b |u(x)|^p w(x)dx < \infty$$

where the integral is understood in the sense of Lebesgue.

II. PROPERTIES OF THE EIGENVALUE EQUATION, KNOWN RESULTS

A. The gauge mode

For $\lambda = 1$ there is an analytic solution of eq. (3) which can be given in closed form, the so–called gauge mode (cf. Ref. 3)

$$\theta(\rho) := \frac{2\rho}{1+\rho^2}. \tag{5}$$

This gauge mode leads to an exponentially growing solution $w(\tau,\rho) = e^{\tau}\theta(\rho)$ of eq. (2) which seems to spoil linear stability of f_0. However, this instability is connected to the freedom of choosing the blow up time T when introducing the adapted coordinates (τ,ρ) (cf. Ref. 3). Therefore, we are only interested in stability modulo this gauge freedom. Nevertheless, the existence of this gauge mode will play an essential role in the further analysis.

B. Asymptotic estimates

Eq. (3) has regular singular points at $\rho = 0, 1$. Using Frobenius' method we can derive asymptotic estimates for solutions of eq. (3). Around $\rho = 0$ there exists a regular solution φ_0 and a singular solution ψ_0. Around $\rho = 1$ the situation is similar but more subtle (cf. table I, c is a constant which might also be zero).

ρ	λ	Analytic solution	Non-analytic solution
$\rho \to 0$	any	$\varphi_0 \sim \rho$	$\psi_0 \sim \rho^{-2}$
$\rho \to 1$	$\lambda \notin \mathbb{Z}$	$\varphi_1 \sim 1$	$\psi_1 \sim (1-\rho)^{1-\lambda}$
	$\lambda \in \mathbb{Z},\ \lambda > 1$	$\varphi_1 \sim 1$	$\psi_1 \sim c\log(1-\rho) + (1-\rho)^{1-\lambda}$
	$\lambda \in \mathbb{Z},\ \lambda \leq 1$	$\varphi_1 \sim (1-\rho)^{1-\lambda}$	$\psi_1 \sim c(1-\rho)^{1-\lambda}\log(1-\rho) + 1$

TABLE I Asymptotic estimates for solutions of eq. (3)

Definition 1. For a given $\lambda \in \mathbb{C}$ we denote by $\varphi_0(\cdot, \lambda)$ the solution of eq. (3) which is analytic around $\rho = 0$ and satisfies $\varphi_0'(0, \lambda) = 2$ ($'$ denotes $d/d\rho$). Similarly, by $\varphi_1(\cdot, \lambda)$ we denote the solution of eq. (3) which is analytic around $\rho = 1$ and satisfies $\varphi_1(1, \lambda) = 1$.

If the dependence on λ is not essential we will sometimes omit it in the argument. For instance we will occasionally write f instead of $f(\cdot, \lambda)$ or $f(\rho)$ instead of $f(\rho, \lambda)$ for a function f of ρ and λ.

Due to the location of the singularities we conclude that φ_0 and φ_1 are analytic on $[0, 1)$ and $(0, 1]$, respectively. For $\rho \in (0, 1)$, eq. (3) is perfectly regular (all coefficients are analytic) and well–known theorems on linear ODEs tell us that the solution space is two–dimensional. Since φ_1 and ψ_1 are linearly independent, it follows that there exist constants c_1 and c_2 such that $\varphi_0 = c_1\varphi_1 + c_2\psi_1$ on $(0, 1)$. If $c_2 = 0$ then φ_0 is analytic on $[0, 1]$ and it is therefore an analytic eigenfunction. Moreover, as long as $\mathrm{Re}\lambda > 0$, the analytic eigenfunctions are exactly the regular ones we are interested in.

C. Sturm–Liouville theory

At first glance one might expect Sturm–Liouville theory to answer all questions concerning solutions of eq. (3) since it can be transformed to Sturm–Liouville form (eq. (4)). Unfortunately this is not true. It turns out that the differential operator defined by eq. (4) is symmetric on the weighted Lebesgue space

$$L^2\left((0,1), \frac{d\rho}{(1-\rho^2)^2}\right)$$

but for $\mathrm{Re}\lambda \leq 1$ the solution $\tilde{\varphi}_1(\rho, \lambda) := \rho(1-\rho^2)^{\lambda/2}\varphi_1(\rho, \lambda)$ we are interested in is not an element of this space. Therefore, Sturm–Liouville theory is useless for $\mathrm{Re}\lambda \leq 1$. Nevertheless, for $\mathrm{Re}\lambda > 1$ it can be applied and it follows that there do not exist analytic eigenfunctions for $\mathrm{Re}\lambda > 1$.

III. INTEGRAL EQUATIONS

In what follows we derive integral equations for φ_0 and φ_1. This is straight–forward although sometimes a little tricky.

A. An integral equation for φ_0

We split eq. (3) as

$$u'' + \left(\frac{2}{\rho} - \frac{2\rho}{1-\rho^2}\right)u' - \left(\frac{2\cos(f_0)}{\rho^2(1-\rho^2)} + \frac{2}{1-\rho^2}\right)u = Q_\lambda u \tag{6}$$

where

$$Q_\lambda u(\rho, \lambda) := \frac{1}{1-\rho^2}\left\{2(\lambda-1)\rho u'(\rho, \lambda) + [\lambda(1+\lambda) - 2]u(\rho, \lambda)\right\}.$$

The idea is to interpret eq. (6) as an "inhomogeneous" equation and apply the variations of constants formula. First of all we need a fundamental system for the "homogeneous" equation

$$u'' + \left(\frac{2}{\rho} - \frac{2\rho}{1-\rho^2}\right)u' - \left(\frac{2\cos(f_0)}{\rho^2(1-\rho^2)} + \frac{2}{1-\rho^2}\right)u = 0 \tag{7}$$

Since eq. (7) is exactly eq. (3) with $\lambda = 1$ it is clear that the gauge mode θ solves eq. (7). Another linearly independent solution is

$$\chi(\rho) := \frac{1}{1+\rho^2}\left(\frac{1}{\rho^2} + 6\rho\log\left(\frac{1-\rho}{1+\rho}\right) + 9\right).$$

Therefore $\{\theta, \chi\}$ is a fundamental system for eq. (7). The Wronskian $W(\theta,\chi) := \theta\chi' - \theta'\chi$ of θ and χ is given by

$$W(\theta,\chi)(\rho) = -\frac{6}{\rho^2(1-\rho^2)}.$$

Inspired by the variations of constants formula we consider the integral equation

$$\begin{aligned} u(\rho,\lambda) = \theta(\rho) - \theta(\rho)\int_0^\rho \frac{\chi(\xi)}{W(\theta,\chi)(\xi)} Q_\lambda u(\xi,\lambda)d\xi \\ +\chi(\rho)\int_0^\rho \frac{\theta(\xi)}{W(\theta,\chi)(\xi)} Q_\lambda u(\xi,\lambda)d\xi \end{aligned} \tag{8}$$

for $\rho \in [0,1)$.

First we will show that studying eq. (8) tells us something about solutions of eq. (3).

Lemma 1. *For some $\rho_0 \in (0,1)$ and a given $\lambda \in \mathbb{C}$ let $u \in C^1[0,\rho_0]$ be a solution of eq. (8). Then $u \in C^2[0,\rho_0]$ and u solves eq. (3).*

Proof. For $\rho \in (0,\rho_0]$ the right-hand side of eq. (8) is obviously twice continuously differentiable. By defining $u''(0) := 0$ we obtain $u \in C^2[0,\rho_0]$ which is then by construction (variations of constants formula) also a solution of eq. (3). □

In what follows we show that eq. (8) has a solution u for any $\lambda \in \mathbb{C}$. This solution u is also a solution of eq. (3) and satisfies $u(0,\lambda) = 0$ as well as $u'(0,\lambda) = 2$. Therefore we have $u = \varphi_0$ and eq. (8) is an integral equation for φ_0. The key ingredient is the following Proposition the proof of which will be postponed to appendix A.1.

Proposition 1. *Let $\lambda \in \mathbb{C}$. Then there exists a $\rho_0 \in (0,1)$ such that the integral equation (8) has a solution $u(\cdot,\lambda) \in C^1[0,\rho_0]$.*

Proof. See appendix A.1. □

Lemma 2. *Let $\lambda \in \mathbb{C}$. Then there exists a $\rho_0 \in (0,1)$ such that $\varphi_0(\cdot,\lambda)$ solves eq. (8) on $[0,\rho_0]$.*

Proof. From Proposition 1 we know that there exists a $\rho_0 \in (0,1)$ such that there is a solution $u(\cdot,\lambda) \in C^1[0,\rho_0]$ of eq. (8). Using Lemma 1 we conclude that $u(\cdot,\lambda)$ is in fact twice continuously differentiable and solves eq. (3). From eq. (8) we observe that $u \neq 0$, $u(0,\lambda) = 0$, $u'(0,\lambda) = 2$. On $(0,1)$ the space of solutions of eq. (3) is spanned by φ_0 and ψ_0 where ψ_0 is a solution of eq. (3) which is singular at $\rho = 0$. Therefore we have $u = c_1\varphi_0 + c_2\psi_0$ for constants c_1 and c_2 on $(0,1)$. The conditions $\lim_{\rho\to 0} u(\rho,\lambda) = 0$ and $\lim_{\rho\to 0} u'(\rho,\lambda) = 2$ yield $c_2 = 0$ and $c_1 = 1$ and thus $\varphi_0 = u$. □

As a last step we claim that φ_0 actually satisfies the integral equation (8) on any $[0,a] \subset [0,1)$ and not only on $[0,\rho_0]$ for the special ρ_0 from Lemma 2.

Proposition 2. *Let $[0,a] \subset [0,1)$ and $\lambda \in \mathbb{C}$. Then the function $\varphi_0(\cdot,\lambda)$ satisfies eq. (8) for all $\rho \in [0,a]$.*

Proof. Let $\rho_0 \in (0,1)$ be the constant from Lemma 2 and choose $a \in (\rho_0, 1)$. Since eq. (3) is regular on $[\rho_0, a]$ everything follows from well-known ODE theory. □

B. An integral equation for φ_1

We deduce a similar integral equation for φ_1. However, due to the singularity of $Q_\lambda u$ at $\rho = 1$ we have to split eq. (3) in a different way. We write

$$u'' + \left(\frac{2}{\rho} - \frac{2\lambda\rho}{1-\rho^2}\right) u' = qu$$

where

$$q(\rho, \lambda) := \frac{\lambda(1+\lambda)}{1-\rho^2} + \frac{2\cos(2f_0)}{\rho^2(1-\rho^2)}.$$

$\{1, \psi\}$ is a fundamental system for the "homogeneous" equation where

$$\psi(\rho, \lambda) := \int_c^\rho \frac{d\xi}{\xi^2(1-\xi^2)^\lambda}$$

and $c \in (0,1)$ is arbitrary. The Wronskian is simply given by $W(1, \psi) = -\psi'$. Thus we consider the integral equation

$$\begin{aligned} u(\rho, \lambda) = 1 &- \int_\rho^1 \frac{\psi(\xi, \lambda)}{\psi'(\xi, \lambda)} q(\xi, \lambda) u(\xi, \lambda) d\xi \\ &+ \psi(\rho, \lambda) \int_\rho^1 \frac{1}{\psi'(\xi, \lambda)} q(\xi, \lambda) u(\xi, \lambda) d\xi \end{aligned} \tag{9}$$

for $\rho \in (0, 1]$. To ensure existence of the integrals we have to restrict ourselves to $\mathrm{Re}\lambda > 0$ which will be assumed from now on.

Now we proceed similarly to the last section with the sole difference that it is sufficient to consider continuous solutions of eq. (9) since no derivative of u appears in eq. (9).

Lemma 3. *For some $\rho_1 \in (0,1)$ and a given $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > 0$ let $u \in C[\rho_1, 1]$ be a solution of eq. (9). Then $u \in C^2[\rho_1, 1]$ and u solves eq. (3).*

Proof. Similar to the proof of Lemma 1. $\square$

Proposition 3. *For a given $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > 0$ there exists a $\rho_1 \in (0,1)$ such that the integral equation (9) has a solution $u(\cdot, \lambda) \in C[\rho_1, 1]$.*

Proof. See appendix A.2. $\square$

Lemma 4. *Let $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > 0$. Then there exists a $\rho_1 \in (0,1)$ such that $\varphi_1(\cdot, \lambda)$ solves eq. (9) on $[\rho_1, 1]$.*

Proof. Similar to the proof of Lemma 2. $\square$

IV. PROPERTIES OF EIGENFUNCTIONS

We derive some properties of analytic eigenfunctions. We will show that (real) eigenfunctions do not have zeros on $(0,1)$ and that the derivative has to change sign on $(0,1)$ if $\lambda > -2$. The first result is established by using a well–known method from classic oscillation theory.

Lemma 5. *For $\lambda \in \mathbb{R}$ let u be a real solution of eq. (3) on $[0,1)$ satisfying $u(0) = 0$ and $u'(0) > 0$. Then $u(\rho) > 0$ for all $\rho \in (0,1)$.*

Proof. Since the Lemma is obviously true for $\lambda = 1$ we assume $\lambda \neq 1$. We argue by contradiction. Suppose u is a solution of eq. (3) with $u(0) = 0$, $u'(0) > 0$ and let $\rho_0 \in (0,1)$ be the smallest zero of u. Thus we have $u(\rho) > 0$ for all $\rho \in (0, \rho_0)$. We define a new dependent variable $\tilde{u}$ by $\tilde{u}(\rho) := \rho(1-\rho^2)^{\lambda/2} u(\rho)$. Then we have $\tilde{u}(0) = 0$, $\tilde{u}(\rho_0) = 0$, $\tilde{u}(\rho) > 0$ for all $\rho \in (0, \rho_0)$ and $\tilde{u}$ satisfies the equation

$$\tilde{u}'' + q\tilde{u} = p_\lambda \tilde{u} \tag{10}$$

where

$$q(\rho) = -\frac{2\cos(2f_0)}{\rho^2(1-\rho^2)}$$

and

$$p_\lambda(\rho) = -\frac{\lambda(2-\lambda)}{(1-\rho^2)^2}.$$

Observe that $p_\lambda - p_1 > 0$ for all $\lambda \in \mathbb{R}$, $\lambda \neq 1$. Now let $\tilde{\theta}(\rho) := \rho\sqrt{(1-\rho^2)}\theta(\rho)$ where θ is the gauge mode (eq. 5). Then $\tilde{\theta}$ satisfies eq. (10) with $\lambda = 1$. Integration by parts yields

$$\int_a^b \left[p_\lambda(\rho) - p_1(\rho)\right] \tilde{u}(\rho)\tilde{\theta}(\rho) d\rho = W(\tilde{u}, \tilde{\theta})(a) - W(\tilde{u}, \tilde{\theta})(b)$$

for $a, b \in [0,1)$. Now choose $a = 0$ and $b = \rho_0$ to obtain

$$\int_0^{\rho_0} \left[p_\lambda(\rho) - p_1(\rho)\right] \tilde{u}(\rho)\tilde{\theta}(\rho) d\rho = \tilde{u}'(\rho_0)\tilde{\theta}(\rho_0).$$

But this is a contradiction since the left-hand side is positive while the right-hand side is negative or equals zero ($\tilde{u}'(\rho_0) \leq 0$ since $\tilde{u}(\rho) > 0$ for all $\rho \in (0, \rho_0)$). □

For the next result we make use of the fact that φ_0 is a solution of eq. (8) on any $[0, a] \subset [0, 1)$.

Lemma 6. *Let $u \in C^2[0,1]$ be a nontrivial solution of eq. (3) for some $\lambda \in \mathbb{C}$. Then u satisfies*

$$\int_0^1 \frac{\theta(\xi)}{W(\theta,\chi)(\xi)} Q_\lambda u(\xi, \lambda) d\xi = 0.$$

Proof. For $\lambda = 1$ the Lemma is trivial, so assume $\lambda \neq 1$. Since u is regular around $\rho = 0$ and not identically zero it follows that there exists a constant c such that $u = c\varphi_0$. According to Proposition 2, u satisfies

$$\begin{aligned} u(\rho, \lambda) = c\theta(\rho) - \theta(\rho) \int_0^\rho \frac{\chi(\xi)}{W(\theta,\chi)(\xi)} Q_\lambda u(\xi, \lambda) d\xi \\ + \chi(\rho) \int_0^\rho \frac{\theta(\xi)}{W(\theta,\chi)(\xi)} Q_\lambda u(\xi, \lambda) d\xi \end{aligned} \tag{11}$$

for all $\rho \in [0,1)$. Now suppose

$$\int_0^1 \frac{\theta(\xi)}{W(\theta,\chi)(\xi)} Q_\lambda u(\xi, \lambda) d\xi \neq 0.$$

Then passing to the limit $\rho \to 1$ in eq. (11) yields divergence of $u(\rho, \lambda)$ since χ diverges at $\rho = 1$ and

$$\frac{\chi}{W(\theta,\chi)} Q_\lambda u(\cdot, \lambda) \in L^1(0,1).$$

However, this is a contradiction to $u \in C^2[0,1]$. □

With these preparations we show that the derivative of regular solutions of eq. (3) has to change sign if $\lambda > -2$.

Lemma 7. *For* $\lambda > -2$, $\lambda \neq 1$ *let* $u \in C^2[0,1]$ *be a nontrivial real solution of eq. (3). Then* u' *changes its sign on* $(0,1)$.

Proof. Suppose u' does not change sign on $(0,1)$. Without loss of generality we assume $u'(\rho) \geq 0$ for all $\rho \in [0,1)$. It follows that $u(\rho) > 0$ for all $\rho \in (0,1)$ ($u(0) = 0$). By Lemma 6, u satisfies

$$\int_0^1 \frac{\theta(\rho)}{W(\theta,\chi)(\rho)} \frac{1}{1-\rho^2} \left\{2(\lambda-1)\rho u'(\rho) + [\lambda(1+\lambda)-2]u(\rho)\right\} d\rho = 0. \tag{12}$$

But since $\operatorname{sgn}(\lambda - 1) = \operatorname{sgn}(\lambda(1+\lambda) - 2)$ the integrand in eq. (12) is either positive or negative on $(0,1)$ which is a contradiction. □

V. PROOF OF THE MAIN RESULT

Now we are able to prove our main result.

Proof of Theorem 1. Suppose $u \in C^2[0,1]$ is a nontrivial solution of eq. (3). Without loss of generality we assume u to be real and $u'(0) > 0$. By Lemma 5 and the asymptotic estimates for φ_1 we observe that $u(1) > 0$. Furthermore, u has to satisfy the regularity condition

$$u'(1) = \frac{2-\lambda-\lambda^2}{2\lambda} u(1)$$

which follows from the existence of $u''(1)$. This means that $u(1)$ and $u'(1)$ have the same sign ($\lambda \in (0,1)$) and therefore $u'(1) > 0$. By Lemma 7 we know that u has a maximum on $(0,1)$, say at $\rho_0 \in (0,1)$. We have $u(\rho_0) > 0$, $u'(\rho_0) = 0$ and $u''(\rho_0) < 0$. Inserting in eq. (3) yields

$$u''(\rho_0) = \beta_\lambda(\rho_0) u(\rho_0) < 0$$

where

$$\beta_\lambda(\rho) := \frac{\lambda(1+\lambda)}{1-\rho^2} + \frac{2\cos(f_0(\rho))}{\rho^2(1-\rho^2)}.$$

Note that β_λ is positive for small ρ and has exactly one zero on $(0,1)$, say at $\rho_\lambda^* \in (0,1)$. It follows that $\rho_0 > \rho_\lambda^*$. Since $u'(1) > 0$, u' has to change sign again, say at $\rho_1 \in (\rho_0, 1)$. Therefore we have $u(\rho_1) > 0$, $u'(\rho_1) = 0$ and $u''(\rho_1) > 0$. Eq. (3) yields

$$u''(\rho_1) = \beta_\lambda(\rho_1) u(\rho_1) > 0$$

which is, however, impossible since $\beta_\lambda(\rho_1) < 0$. □

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APPENDIX A: Existence of solutions of integral equations

1. Proof of Proposition 1

Proof of Proposition 1. Define the mapping $K : C^1[0,\rho_0] \to C^1[0,\rho_0]$ by

$$Ku(\rho) := \theta(\rho) - \theta(\rho)\int_0^\rho \frac{\chi(\xi)}{W(\theta,\chi)(\xi)} Q_\lambda u(\xi)d\xi + \chi(\rho)\int_0^\rho \frac{\theta(\xi)}{W(\theta,\chi)(\xi)} Q_\lambda u(\xi)d\xi$$

where $\rho_0 \in (0,1)$ is to be chosen later. Differentiation yields

$$(Ku)'(\rho) = \theta'(\rho) - \theta'(\rho)\int_0^\rho \frac{\chi(\xi)}{W(\theta,\chi)(\xi)} Q_\lambda u(\xi)d\xi + \chi'(\rho)\int_0^\rho \frac{\theta(\xi)}{W(\theta,\chi)(\xi)} Q_\lambda u(\xi)d\xi.$$

For $u, v \in C^1[0,\rho_0]$ we readily estimate

$$\|Ku - Kv\|_{C^1[0,\rho_0]} \leq C\|\alpha + \beta\|_{C[0,\rho_0]}\|u - v\|_{C^1[0,\rho_0]}$$

where

$$\alpha(\rho) := |\theta(\rho)|\int_0^\rho \frac{|\chi(\xi)|}{(1-\xi^2)|W(\theta,\chi)(\xi)|}d\xi + |\chi(\rho)|\int_0^\rho \frac{|\theta(\xi)|}{(1-\xi^2)|W(\theta,\chi)(\xi)|}d\xi$$

and

$$\beta(\rho) := |\theta'(\rho)|\int_0^\rho \frac{|\chi(\xi)|}{(1-\xi^2)|W(\theta,\chi)(\xi)|}d\xi + |\chi'(\rho)|\int_0^\rho \frac{|\theta(\xi)|}{(1-\xi^2)|W(\theta,\chi)(\xi)|}d\xi$$

and $C > 0$ is a constant. Using de l'Hospital's rule one readily shows that $\lim_{\rho\to 0}(\alpha(\rho) + \beta(\rho)) = 0$ and therefore we can choose $\rho_0 > 0$ so small that $C' := C\|\alpha + \beta\|_{C[0,\rho_0]} < 1$ and K satisfies the contraction property. Hence, existence of a fixed point of K follows from the contraction mapping principle on the Banach space $C^1[0,\rho_0]$. □

2. Proof of Proposition 3

Proof of Proposition 3. For fixed $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > 0$ we define the mapping $K : C[\rho_1, 1] \to C[\rho_1, 1]$ by

$$Ku(\rho) := 1 - \int_\rho^1 \frac{\psi(\xi,\lambda)}{\psi'(\xi,\lambda)} q(\xi,\lambda)u(\xi)d\xi + \psi(\rho,\lambda)\int_\rho^1 \frac{1}{\psi'(\xi,\lambda)} q(\xi,\lambda)u(\xi)d\xi.$$

For $u, v \in C[\rho_1, 1]$ we estimate

$$\|Ku - Kv\|_{C[\rho_1,1]} \leq \|\alpha(\cdot,\lambda)\|_{C[\rho_1,1]}\|u - v\|_{C[\rho_1,1]}$$

where

$$\alpha(\rho,\lambda) = \int_\rho^1 \left|\frac{\psi(\xi,\lambda)}{\psi'(\xi,\lambda)} q(\xi,\lambda)\right| d\xi + |\psi(\rho,\lambda)|\int_\rho^1 \left|\frac{q(\xi,\lambda)}{\psi'(\xi,\lambda)}\right| d\xi.$$

De l'Hospital's rule implies that $\lim_{\rho\to 1}\alpha(\rho,\lambda) = 0$ and thus, for ρ_1 sufficiently close to 1, the contraction mapping principle on $C[\rho_1, 1]$ guarantees the existence of a fixed point of K. □

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